Cauchy process on half-line

Mateusz Kwaśnicki

Institute of Mathematics, Wrocław University of Technology

Let X_t be the one-dimensional Cauchy process. There are various characterizations of X_t: it is the Lévy process with no drift, no Gaussian part and Lévy measure $\pi^{-1}x^{-2}dx$; X_t is the symmetric 1-stable process; X_t can be constructed as the Brownian motion subordinated by $\frac{1}{2}$-stable subordinator; (X_t) also describes visits of 2-dimensional Brownian motion on a line (more formally, it is 2-dimensional Brownian motion seen at the inverse local time of a line). The transition density of the Cauchy process is given by the Cauchy distribution,

$$\mathbf{P}_x(X_t \in dy) = \frac{1}{\pi} \frac{t}{t^2 + (y-x)^2} \, dy, \qquad \mathbf{E}_x e^{i\xi X_t} = e^{i\xi x - t|\xi|};$$

as usual, $\mathbf{P}_x$ and $\mathbf{E}_x$ correspond to the process starting at x. I will present the results of my recent work with Tadeusz Kulczycki, Jacek Małecki and Andrzej Stós concerning the Cauchy process killed upon hitting $(-\infty, 0]$.

Let $p_t^+(x, \cdot)$ be the density of the (sub-probabilistic) distribution of X_t starting at x and killed when it hits $(-\infty, 0]$,

$$p_t^+(x,y)dy = \mathbf{P}_x\big(X_t \in dy \; ; \; X_s > 0 \text{ for } s \in [0,t]\big).$$

Then $p_t^+(x,y)$ $(x, y, t > 0)$ is the kernel of the semi-group of contraction operators P_t^+ acting on an appropriate function space, e.g. the space $C(\mathbf{R}_+)$ of bounded continuous functions on $(0, \infty)$, or $L^2(\mathbf{R}_+)$.

The spectrum of P_t^+ on $L^2(\mathbf{R}_+)$ is continuous. I will show, however, that for all $\lambda > 0$ there is a function $\psi_\lambda \in C(\mathbf{R}_+)$ satisfying $P_t^+ \psi_\lambda = e^{-\lambda t}\psi_\lambda$. Furthermore, ψ_λ is given by an explicit formula,

$$\psi_\lambda(x) = \sin(\lambda x + \tfrac{\pi}{8}) - \frac{\sqrt{2}}{2\pi} \int_0^\infty \frac{t}{1+t^2} \exp\left(-\frac{1}{\pi} \int_0^\infty \frac{\log(t+s)}{1+s^2} ds \right) e^{-t\lambda x} dt.$$

These functions are used to define the integral transform

$$\Pi f(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \psi_\lambda(x) dx.$$

This gives a spectral representation of P_t^+, that is, Π is the unitary mapping of $L^2(\mathbf{R}_+)$ into $L^2(\mathbf{R}_+)$ and $\Pi P_t^+ f(\lambda) = e^{-\lambda t} \Pi f(\lambda)$. A version of Plancherel's theorem and inversion formula also holds true for Π.

The picture is therefore very similar to that of the Brownian motion. For the Brownian motion killed upon hitting $(-\infty, 0]$, the eigenfunctions are simply sines $\sin(\lambda x)$, and the corresponding integral transform is simply the Fourier sine transform on $(0, \infty)$.

The derivation of the explicit formula for ψ_λ uses the identification of the spectral problem for P_t^+ and a two-dimensional spectral problem for the Laplace operator with spectral parameter on the boundary, the so-called mixed Steklov problem. This relation was shown recently by Bañuelos and Kulczycki [1]. A closely related sloshing problem with semi-infinite dock has been solved in 1947 by Friedrichs and Lewy [4] (see also [5]), and their methods can be adapted to our setting.

Applications of the above results include the explicit formula for $p_t^+(x, y)$. To my knowledge, this is the first explicit formula for the transition density of a killed stable process other than Brownian motion, and only two-sided estimates of Chen, Kim and Song [3] were available (see also [2]). Also, the distribution of the hitting time $\tau = \inf\{t \geq 0 \, : \, X_t \leq 0\}$ is given by

$$\mathbf{P}^x(\tau \in dt) = \frac{1}{\pi} \frac{x}{t^2 + x^2} \exp\left(\frac{1}{\pi} \int_0^\infty \frac{\log(\frac{t}{x} + s)}{1 + s^2} \, ds \right) dt.$$

This can be obtained either by integration of $p_t^+(x, y)$ or independently, by solving a problem similar to mixed Steklov problem for ψ_λ.

References

[1] R. Bañuelos, T. Kulczycki, *The Cauchy process and the Steklov problem.* J. Funct. Anal. 211(2) (2004), pp. 355–423.

[2] K. Bogdan, T. Grzywny, *Heat kernel of fractional Laplacian in cones.* Preprint

[3] Z.Q. Chen, P. Kim, R. Song, *Heat kernel estimates for Dirichlet fractional Laplacian.* To appear in J. Eur. Math. Soc.

[4] K.O. Friedrichs, H. Lewy, *The dock problem.* Commun. Pure Appl. Math. 1 (1948), pp. 135–148.

[5] R.L. Holford, *Short surface waves in the presence of a finite dock. I, II.* Proc. Cambridge Philos. Soc. 60 (1964), pp. 957–983, 985–1011.